Alternative evaluation of a ln tan integral arising in quantum field theory

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Abstract

A certain dilogarithmic integral I_7 turns up in a number of contexts including Feynman diagram calculations, volumes of tetrahedra in hyperbolic geometry, knot theory, and conjectured relations in analytic number theory. We provide an alternative explicit evaluation of a parameterized family of integrals containing this particular case. By invoking the Bloch-Wigner form of the dilogarithm function, we produce an equivalent result, giving a third evaluation of I_7. We also alternatively formulate some conjectures which we pose in terms of values of the specific Clausen function Cl_2.

Key words and phrases

Clausen function, dilogarithm function, Hurwitz zeta function, functional equation, duplication formula, triplication formula

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The particular integral

$$I_7 \equiv \frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \ln \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt, \tag{1}$$

occurs in a number of contexts and has received significant attention in the last several years [3, 4, 5, 6]. This and related integrals arise in hyperbolic geometry, knot theory, and quantum field theory [6, 7, 8]. Very recently [9] we obtained an explicit evaluation of (1) in terms of the specific Clausen function Cl_2. However, much work remains. This is due to the conjectured relation between a Dirichlet L series and I_7 [6],

$$I_7 \stackrel{?}{=} L_{-7}(2) = \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right].$$
$$\tag{2}$$

The ? here indicates that numerical verification to high precision has been performed but that no proof exists, the approximate numerical value of I_7 being $I_7 \simeq$ 1.15192547054449104710169. The statement (2) is equivalent to the conjecture, with $\theta_7 \equiv 2 \tan^{-1} \sqrt{7}$,

$$\frac{1}{2} \left[3 Cl_2(\theta_7) - 3 Cl_2(2\theta_7) + Cl_2(3\theta_7) \right] \stackrel{?}{=} \frac{1}{4} Z_{Q(\sqrt{-7})} = \frac{7}{4} \left[Cl_2\left(\frac{2\pi}{7}\right) + Cl_2\left(\frac{4\pi}{7}\right) - Cl_2\left(\frac{6\pi}{7}\right) \right],$$
$$\tag{3}$$

relating triples of Clausen function values. Here, we alternatively evaluate I_7 directly in terms of the left side of (3). In addition, we present another evaluation of I_7, based upon a property of the Bloch-Wigner form of the dilogarithm function.

We recall that the L series $L_{-7}(s)$ has occurred in several places before, including hyperbolic geometry [19] and Dedekind sums of analytic number theory [2]. Let $\zeta_{Q(\sqrt{-p})}$ denote the Dedekind zeta function of an imaginary quadratic field $Q(\sqrt{-p})$.

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Then indeed we have [2, 19, 20]

$$\zeta_{Q(\sqrt{-7})}(s) = \frac{1}{2} \sum_{\substack{m,n \in Z \\ (m,n) \neq (0,0)}} \frac{1}{(m^2 + mn + 2n^2)^s} \tag{4}$$

$$= \zeta(s)L_{-7}(s) = \zeta(s)7^{-s} \sum_{\nu=1}^{6} \left(\frac{\nu}{7}\right) \zeta\left(s, \frac{\nu}{7}\right), \tag{5}$$

where $\left(\frac{\nu}{7}\right)$ is a Legendre symbol, $\zeta(s, a)$ is the Hurwitz zeta function, and $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function.

The series $L_{-7}(s)$ is an example of a Dirichlet L function corresponding to a real character χ_k [here, modulo 7] with $\chi_k(k - 1) = -1$. Such L functions, extendable to the whole complex plane, satisfy the functional equation [20]

$$L_{-k}(s) = \frac{1}{\pi}(2\pi)^s k^{-s+1/2} \cos\left(\frac{s\pi}{2}\right) \Gamma(1 - s)L_{-k}(1 - s). \tag{6}$$

Owing to the relation $\Gamma(1 - s)\Gamma(s) = \pi/\sin(\pi s)$, this functional equation may also be written in the form

$$L_{-k}(1 - s) = 2(2\pi)^{-s} k^{s-1/2} \sin\left(\frac{\pi s}{2}\right) \Gamma(s)L_{-k}(s). \tag{7}$$

Integral representations are known for these L-functions [20, 10]. From the functional equation (6) we find

$$\left.\frac{\partial}{\partial s}L_{-k}(s)\right|_{s=-1} = \frac{k^{3/2}}{4\pi}L_{-k}(2). \tag{8}$$

In turn, we have

$$\zeta'_{Q(\sqrt{-k})}(-1) = -\frac{k^{3/2}}{48\pi}L_{-k}(2), \tag{9}$$

where we used $\zeta(-1) = -1/12$ and $L_{-k}(-1) = 0$.

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We have

Proposition 1. We have

$$I_7 = \frac{4}{7\sqrt{7}} \left[3\mathrm{Cl}_2(\theta_7) - 3\mathrm{Cl}_2(2\theta_7) + \mathrm{Cl}_2(3\theta_7) \right]. \tag{10}$$

In fact, we treat integrals

$$I(a) \equiv \int_{\pi/3}^{\pi/2} \ln \left| \frac{\tan t + a}{\tan t - a} \right| dt, \tag{11}$$

and more general ones with varying limits. For (11), we assume that $\pi/3 < \varphi = \tan^{-1} a < \pi/2$. These other integrals permit us to explicitly write other conjectures directly in terms of linear combinations of specific Clausen function values.

The Clausen function Cl_2 can be defined by (e.g., [14, 16])

$$\mathrm{Cl}_2(\theta) \equiv -\int_0^\theta \ln \left| 2 \sin \frac{t}{2} \right| dt = \int_0^1 \tan^{-1} \left(\frac{x \sin \theta}{1 - x \cos \theta} \right) \frac{dx}{x} \tag{12}$$

$$= -\sin \theta \int_0^1 \frac{\ln x}{x^2 - 2x \cos \theta + 1} dx = \sum_{n=1}^\infty \frac{\sin(n\theta)}{n^2}. \tag{13}$$

When θ is a rational multiple of π it is known that $\mathrm{Cl}_2(\theta)$ may be written in terms of the trigamma and sine functions [11, 13]. This Clausen function is odd and periodic, $\mathrm{Cl}_2(\theta) = -\mathrm{Cl}_2(-\theta)$, and $\mathrm{Cl}_2(\theta) = \mathrm{Cl}_2(\theta + 2\pi)$. It also satisfies the duplication

$$\frac{1}{2}\mathrm{Cl}_2(2\theta) = \mathrm{Cl}_2(\theta) - \mathrm{Cl}_2(\pi - \theta), \tag{14}$$

triplication

$$\frac{1}{3}\mathrm{Cl}_2(3\theta) = \mathrm{Cl}_2(\theta) + \mathrm{Cl}_2\left(\theta + \frac{2\pi}{3}\right) + \mathrm{Cl}_2\left(\theta + \frac{4\pi}{3}\right), \tag{15}$$

and quadriplication

$$\frac{1}{4}\mathrm{Cl}_2(4\theta) = \mathrm{Cl}_2(\theta) + \mathrm{Cl}_2\left(\theta + \frac{\pi}{2}\right) + \mathrm{Cl}_2\left(\theta + \pi\right) + \mathrm{Cl}_2\left(\theta + \frac{3\pi}{2}\right), \tag{16}$$

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formulas, as well as a more general multiplication formula [14]. We recall the specific relation

$$\sum_{j=1}^{6} \text{Cl}_2\left(\frac{2\pi}{7}j\right) = 0, \tag{17}$$

that arises as a special case of [14] (pp. 95, 253)

$$\sum_{j=1}^{n-1} \text{Cl}_2\left(\frac{2\pi}{n}j\right) = 0. \tag{18}$$

In (17), pairwise cancellation occurs, as $\text{Cl}_2(\theta) = -\text{Cl}_2(2\pi - \theta)$.

Further information on the special functions that we employ may readily be found elsewhere [15, 16, 18, 10]. In particular, with

$$\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \qquad |z| \le 1, \tag{19}$$

or

$$\text{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt, \tag{20}$$

the dilogarithm function, we have the relation

$$\text{Li}_2(e^{i\theta}) = \frac{\pi^2}{6} - \frac{1}{4}\theta(2\pi - \theta) + i\text{Cl}_2(\theta), \qquad 0 \le \theta \le 2\pi. \tag{21}$$

We omit discussion of further relations between the Clausen function Cl_2 and the dilogarithm function.

For the proof of Proposition 1 we repeatedly rely on [14] (pp. 227, 272)

$$\int_0^\theta \ln(\tan\theta + \tan\varphi)d\theta = -\theta\ln(\cos\varphi) - \frac{1}{2}\text{Cl}_2(2\theta + 2\varphi) + \frac{1}{2}\text{Cl}_2(2\varphi) - \frac{1}{2}\text{Cl}_2(\pi - 2\theta). \tag{22}$$

We may split the integral in (11), writing

$$I(a) = \int_{\pi/3}^{\varphi} \ln\left(\frac{a + \tan t}{a - \tan t}\right) dt + \int_{\varphi}^{\pi/2} \ln\left(\frac{\tan t + a}{\tan t - a}\right) dt$$

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$$= \int_{\pi/3}^{\pi/2} \ln(a + \tan t)dt - \int_{\pi/3}^{\varphi} \ln(a - \tan t)dt - \int_{\varphi}^{\pi/2} \ln(\tan t - a)dt. \qquad (23)$$

By the use of (22) we obtain for the first integral on the right side of (23)

$$\int_{\pi/3}^{\pi/2} \ln(a + \tan t)dt = -\frac{\pi}{6} \ln \cos \varphi + \frac{1}{2} \left[\mathrm{Cl}_2\left(\frac{2\pi}{3} + 2\varphi \right) - \mathrm{Cl}_2\left(\pi + 2\varphi \right) \right] + \frac{1}{2} \mathrm{Cl}_2\left(\frac{\pi}{3} \right),$$
$$(24a)$$

the second integral,

$$\int_{\pi/3}^{\varphi} \ln(a - \tan t)dt = -\left(\varphi - \frac{\pi}{3} \right) \ln \cos \varphi - \frac{1}{2}\mathrm{Cl}_2\left[2\left(\varphi - \frac{\pi}{3} \right) \right] + \frac{1}{2}\left[\mathrm{Cl}_2\left(\frac{\pi}{3} \right) - \mathrm{Cl}_2(\pi - 2\varphi) \right],$$
$$(24b)$$

and the third integral,

$$\int_{\varphi}^{\pi/2} \ln(\tan t - a)dt = -\left(\frac{\pi}{2} - \varphi \right) \ln \cos \varphi = -\cot^{-1} a \ln \cos \varphi. \qquad (24c)$$

This latter integral is readily obtained from (22) by taking $a \to -a$ so that simply $\tan \varphi \to -\tan \varphi$. Then per (23) we have

$$I(a) = \frac{1}{2} \left[\mathrm{Cl}_2\left(2\varphi + \frac{2\pi}{3} \right) + \mathrm{Cl}_2\left(2\varphi - \frac{2\pi}{3} \right) \right] - \mathrm{Cl}_2(\pi + 2\varphi). \qquad (25)$$

Then we apply both the duplication formula (14) and the triplication formula (15) wherein $\mathrm{Cl}_2(\theta + 4\pi/3) = \mathrm{Cl}_2(\theta - 2\pi/3)$ as $\mathrm{Cl}_2(\theta) = \mathrm{Cl}_2(\theta - 2\pi)$ by the 2π-periodicity of Cl_2. We find

$$I(a) = \frac{1}{6} \left[\mathrm{Cl}_2(6\varphi) - 3\mathrm{Cl}_2(4\varphi) + 3\mathrm{Cl}_2(2\varphi) \right]. \qquad (26)$$

When $\varphi = \tan^{-1} \sqrt{7}$, the case (10) follows.

We next present some reference integrals. We then apply them to write expressions for combinations of the integrals

$$I_n \equiv \int_{n\pi/24}^{(n+1)\pi/24} \ln \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt, \qquad (27)$$

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where $n \geq 0$ is an integer. We supplement (22) with

$$\int_x^y \ln(a - \tan t)dt = -(y - x) \ln \cos \varphi + \frac{1}{2} \left[\text{Cl}_2[2(\varphi - y)] - \text{Cl}_2[2(\varphi - x)] \right]$$

$$-\frac{1}{2} \left[\text{Cl}_2(\pi - 2y) - \text{Cl}_2(\pi - 2x) \right], \quad (28)$$

where $a = \tan \varphi$. We also have

$$\int_x^y \ln \left| \frac{\tan t + a}{\tan t - a} \right| dt = \frac{1}{2} \left[\text{Cl}_2(2x + 2\varphi) - \text{Cl}_2(2x - 2\varphi) + \text{Cl}_2(2y - 2\varphi) - \text{Cl}_2(2y + 2\varphi) \right],$$

$$(29)$$

with $x < \varphi = \tan^{-1} a < y$. We now write expressions for the linear combinations

$$C_1 \equiv -2(I_2 + I_3 + I_4 + I_5) + I_8 + I_9 - (I_{10} + I_{11}) \overset{?}{=} 0, \quad (30)$$

and

$$C_2 \equiv I_2 + 3(I_3 + I_4 + I_5) + 2(I_6 + I_7) - 3I_8 - I_9 \overset{?}{=} 0. \quad (31)$$

These relations have been detected with further PSLQ computations [5]. A similar conjecture for integrals I_n with increments $n\pi/60$ has also been written [4] (p. 508). The latter linear combination may also be expressed in terms of Cl_2 values, but here we concentrate on (30) and (31). We decompose the left side of (30) as indicated, and find for these contributions

$$2(I_2 + I_3 + I_4 + I_5) = \text{Cl}_2\left(\frac{\pi}{6} + 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{6} - 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{2} - 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{2} + 2\varphi\right),$$

$$(32)$$

where $\varphi = \tan^{-1} \sqrt{7}$,

$$2I_8 = \text{Cl}_2\left(\frac{2\pi}{3} + 2\varphi\right) - \text{Cl}_2\left(\frac{3\pi}{4} + 2\varphi\right) + \text{Cl}_2\left(2\varphi - \frac{2\pi}{3}\right) - \text{Cl}_2\left(2\varphi - \frac{3\pi}{4}\right), \quad (33)$$

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$$2I_9 = \text{Cl}_2\left(\frac{3\pi}{4} + 2\varphi\right) - \text{Cl}_2\left(\frac{3\pi}{4} - 2\varphi\right) + \text{Cl}_2\left(\frac{5\pi}{6} - 2\varphi\right) - \text{Cl}_2\left(\frac{5\pi}{6} + 2\varphi\right), \quad (34)$$

and

$$I_{10} + I_{11} = -\text{Cl}_2(\pi + 2\varphi) + \frac{1}{2}\left[\text{Cl}_2\left(\frac{5\pi}{6} + 2\varphi\right) - \text{Cl}_2\left(\frac{5\pi}{6} - 2\varphi\right)\right]. \quad (35)$$

Therefore, we obtain

$$C_1 = -\text{Cl}_2\left(\frac{\pi}{6} + 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{6} - 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{2} - 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{2} + 2\varphi\right)$$

$$\frac{1}{2}\left[\text{Cl}_2\left(\frac{2\pi}{3} + 2\varphi\right) + \text{Cl}_2\left(2\varphi - \frac{2\pi}{3}\right)\right] + \text{Cl}_2\left(\frac{5\pi}{6} - 2\varphi\right) - \text{Cl}_2\left(\frac{5\pi}{6} + 2\varphi\right)$$

$$+\text{Cl}_2(\pi + 2\varphi). \quad (36)$$

For the combination C_2 we have

$$2I_2 = \text{Cl}_2\left(\frac{\pi}{6} + 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{6} - 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{4} - 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{4} + 2\varphi\right), \quad (37)$$

$$-2(I_6+I_7) = \text{Cl}_2\left(\frac{2\pi}{3} + 2\varphi\right) - \text{Cl}_2\left(\frac{2\pi}{3} - 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{2} - 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{2} + 2\varphi\right), \quad (38)$$

and

$$2(I_3+I_4+I_5) = \text{Cl}_2\left(\frac{\pi}{4} + 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{4} - 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{2} - 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{2} + 2\varphi\right). \quad (39)$$

Therefore, we find

$$2C_2 = \text{Cl}_2\left(\frac{\pi}{6} + 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{6} - 2\varphi\right) + \text{Cl}_2\left(\frac{\pi}{2} - 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{2} + 2\varphi\right)$$

$$-5\text{Cl}_2\left(\frac{2\pi}{3} + 2\varphi\right) - 5\text{Cl}_2\left(2\varphi - \frac{2\pi}{3}\right) - \text{Cl}_2\left(\frac{5\pi}{6} - 2\varphi\right) + \text{Cl}_2\left(\frac{5\pi}{6} + 2\varphi\right)$$

$$+2\left[\text{Cl}_2\left(\frac{\pi}{4} + 2\varphi\right) - \text{Cl}_2\left(\frac{\pi}{4} - 2\varphi\right) + \text{Cl}_2\left(\frac{3\pi}{4} + 2\varphi\right) + \text{Cl}_2\left(2\varphi - \frac{3\pi}{4}\right)\right]. \quad (40)$$

By a combination of the quadriplication formula (16) and the duplication formula (14) we may write

$$\frac{1}{4}\text{Cl}_2(4\theta) = \text{Cl}_2\left(\theta + \frac{\pi}{2}\right) + \text{Cl}_2\left(\theta - \frac{\pi}{2}\right) + \frac{1}{2}\text{Cl}_2(2\theta). \tag{41}$$

This enables other expressions for C_1 and C_2. Similarly, one may use the 6- and 12-fold multiplication formulas.

In regard to the combination on the right side of (3), we comment on an observation given previously [9]. We have that $\pm\sin(2\pi/7)$, $\pm\sin(4\pi/7)$, and $\pm\sin(6\pi/7)$ are the nonzero roots of the Chebyshev polynomial $T_7(x)$. Indeed, if we write the cubic polynomials

$$p_1(x) = \left(x - \sin\frac{2\pi}{7}\right)\left(x - \sin\frac{4\pi}{7}\right)\left(x + \sin\frac{6\pi}{7}\right) = x^3 - \frac{\sqrt{7}}{2}x^2 + \frac{\sqrt{7}}{8}, \tag{42a}$$

and

$$p_2(x) = \left(x - \sin\frac{6\pi}{7}\right)\left(x + \sin\frac{2\pi}{7}\right)\left(x + \sin\frac{4\pi}{7}\right) = x^3 + \frac{\sqrt{7}}{2}x^2 - \frac{\sqrt{7}}{8}, \tag{42b}$$

we then have the factorization $p_1(x)p_2(x) = T_7(x)/64x$. This invites questions as to whether scaled versions of these or other Chebyshev polynomials could be useful in developing identities underlying (3), (30), (31), or the like.

Given the close relation of the Clausen function Cl_2 and the dilogarithm function, one wonders if a set of ladder relations for the latter may be carried over to explain (3) and relations amongst the integrals I_n. In developing ladder relations, cyclotomic equations for the base have proven very useful. It would be of interest to see if Cl_2 relations with θ_7 could be discovered in this way.

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We remark on using Kummer's relation [14] (pp. 107, 254) to rewrite the right side of (3) in terms of the dilogarithm of complex argument. We have

$$\frac{1}{4}Z_{Q(\sqrt{-7})} = \frac{7}{2}\left[\text{Im } \text{Li}_2(Re^{i\phi}) - b\ln R\right], \tag{43}$$

where

$$R = \frac{\tan b}{\sin\phi + \tan b\cos\phi}. \tag{44}$$

Here, we may take $\phi = \pi/7$ and $b = 2\pi/7$, or vice versa. Then by Proposition 2 of [9] we have the integral representation

$$\frac{\sqrt{7}}{2}I_7 \stackrel{?}{=} \text{Cl}_2\left(\frac{2\pi}{7}\right) + \text{Cl}_2\left(\frac{4\pi}{7}\right) - \text{Cl}_2\left(\frac{6\pi}{7}\right) = 2\sin\left(\frac{\pi}{7}\right)\int_d^\infty \frac{\ln y\ dy}{y^2 - 2y\cos(\pi/7) + 1}, \tag{45}$$

where $d = [2\cos(\pi/7) - 1]^{-1}$.

Finally, we use relations from [16] (Appendix A) and [19] to write a third evaluation of the integral I_7. For this we introduce the angle $\theta_{75} \equiv 2\tan^{-1}(\sqrt{7}/5)$ and the Bloch-Wigner dilogarithm [17]

$$D(z) = \text{Im}[\text{Li}_2(z)] + \arg(1-z)\ln|z|, \tag{46}$$

for which we have [16] (p. 246)

$$D(z) = \frac{1}{2}[\text{Cl}_2(2\theta) + \text{Cl}_2(2\omega) - \text{Cl}_2(2\theta + 2\omega)], \tag{47}$$

where $\theta = \arg z$ and $\omega = \arg(1 - \bar{z})$. We note the interpretation that for $z \in C$, the volume of the asymptotic simplex with vertices $0, 1, z$, and ∞ in 3-dimensional

hyperbolic space is given by $|D(z)|$ [16] (p. 271). We then rewrite the expression ([16], p. 384 or [19], p. 246)

$$\zeta_{Q(\sqrt{-7})}(2) = \zeta(2)L_{-7}(2) = \frac{4\pi^2}{21\sqrt{7}} \left[2D\left(\frac{1 + i\sqrt{7}}{2}\right) + D\left(\frac{-1 + i\sqrt{7}}{4}\right) \right]. \tag{48}$$

We apply (47), giving

$$I_7 \overset{?}{=} L_{-7}(2) = \frac{8}{7\sqrt{7}} \left[2D\left(\frac{1 + i\sqrt{7}}{2}\right) + D\left(\frac{-1 + i\sqrt{7}}{4}\right) \right]$$

$$= \frac{4}{7\sqrt{7}} \left[4\mathrm{Cl}_2(\pi - \theta_7) - \mathrm{Cl}_2(\theta_7) + \mathrm{Cl}_2(\theta_{75}) + \mathrm{Cl}_2(\theta_7 - \theta_{75}) \right]. \tag{49}$$

In the case of $D[(1 + i\sqrt{7})/2]$ we used the duplication formula (14). In contrast to (49), the expression in [9] for I_7 involves $\theta_+ \equiv \tan^{-1}(\sqrt{7}/3)$. With the various analytic evaluations now known for I_7 or $L_{-7}(2)$, we have enlarged the set of possible relations amongst Cl_2 values. From (10), (14), and (49) we obtain the conjecture

$$\mathrm{Cl}_2(3\theta_7) - \mathrm{Cl}_2(2\theta_7) \overset{?}{=} \mathrm{Cl}_2(\theta_{75}) + \mathrm{Cl}_2(\theta_7 - \theta_{75}). \tag{50}$$

In fact, we have $\theta_7 - \theta_{75} = 2\theta_+$, and we conclude by proving (50), and thereby (49). We quickly show that both

$$\mathrm{Cl}_2(3\theta_7) = \mathrm{Cl}_2(\theta_{75}) \tag{51}$$

and

$$\mathrm{Cl}_2(2\theta_7) = -\mathrm{Cl}_2(\theta_7 - \theta_{75}), \tag{52}$$

for we have $3\theta_7 - 2\pi = \theta_{75}$ and $\theta_7 - \pi = -\theta_+$. The latter relations require nothing more than the identity $\tan(x/2) = \sin x/(1 + \cos x)$.

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We have similarly found many other angular pairs (θ_1, θ_2) satisfying $3\theta_1 - 2\pi = \pm\theta_2$, immediately giving $\text{Cl}_2(3\theta_1) = \pm\text{Cl}_2(\theta_2)$. As these may be useful elsewhere [6, 16], we record several of them in the first Appendix. We also relegate to this Appendix a possibly new log trigonometric integral in terms of Cl_2. In the second Appendix, we develop new series and integral representations of the Clausen function.

Appendix A

We let $\theta_k \equiv 2\tan^{-1}\sqrt{k}$, and $\theta_{k,j} \equiv 2\tan^{-1}\sqrt{k}/j$. We find the relations

$$3\theta_2 - 2\pi = -\theta_{2,5}, \qquad 3\theta_5 - 2\pi = \theta_{5,7}, \tag{A.1}$$

$$3\theta_{11} - 2\pi = -\theta_{11,4}, \qquad 3\theta_{13} - 2\pi = -2\tan^{-1}\left(\frac{5\sqrt{13}}{19}\right), \tag{A.2}$$

and

$$3\theta_{91,3} - 2\pi = -2\tan^{-1}\left(\frac{8\sqrt{91}}{99}\right), \qquad 3\theta_{91,5} - 2\pi = 2\tan^{-1}\left(\frac{2\sqrt{91}}{155}\right), \qquad 3\theta_{91,7} - 2\pi = -\theta_{91,28}. \tag{A.3}$$

With $\theta_{32} \equiv 2\tan^{-1}\sqrt{3/2}, \theta_{53} \equiv 2\tan^{-1}\sqrt{5/3}, \theta_{133} \equiv 2\tan^{-1}\sqrt{13/3}, \theta_{73} \equiv 2\tan^{-1}\sqrt{7/3}$, we have

$$3\theta_{32} - 2\pi = -2\tan^{-1}\left(\frac{3}{7}\sqrt{\frac{3}{2}}\right), \qquad 3\theta_{53} - 2\pi = -2\tan^{-1}\left(\frac{1}{3}\sqrt{\frac{5}{3}}\right), \tag{A.4}$$

and

$$3\theta_{133} - 2\pi = 2\tan^{-1}\left(\frac{1}{9}\sqrt{\frac{13}{3}}\right), \qquad 3\theta_{73} - 2\pi = -2\tan^{-1}\left(\frac{1}{9}\sqrt{\frac{7}{3}}\right). \tag{A.5}$$

Based upon the trigonometric identity $3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2$, we have found the integral

$$\int_0^x \ln(3 + 4\cos\theta + \cos 2\theta)d\theta = -x\ln 2 + 4\text{Cl}_2(\pi - x), \qquad 0 \le x \le \pi. \tag{A.6}$$

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This obviously provides an integral expression for the Catalan constant $G = \sum_{k \geq 0} (-1)^k / (2k+1)^2 = \text{Cl}_2(\pi/2)$ when $x = \pi/2$.

The function $\text{Cl}_2(t)$ for $t \in (0, \pi)$ has its only maximum at $t = \pi/3$, when $\text{Cl}_2(\pi/3) \simeq 1.014941606409653625021$. We mention that near this value Cl_2 has a fixed point, $\text{Cl}_2(y) = y$ for $y \simeq 1.014471938952517257984144$.

Related to the equality of expressions (10) and (49) for I_7 we have the relation

$$6 \left[\text{Li}_2 \left(\frac{1 - 3i\sqrt{7}}{8} \right) + \text{Li}_2 \left(\frac{1 + 3i\sqrt{7}}{8} \right) \right] = 3(\pi - 2\theta_+)^2 - \pi^2$$

$$= 3(\theta_7 - \theta_+)^2 - \pi^2 = 3[\pi - \tan^{-1}(3\sqrt{7})]^2 - \pi^2. \tag{A.7}$$

Such relations follow readily from (21) as we have

$$6[\text{Li}_2(e^{i\theta}) + \text{Li}_2(e^{-i\theta})] = 2\pi^2 + 3\theta^2, \qquad 0 \leq \theta \leq 2\pi. \tag{A.8}$$

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Appendix B

We have

Proposition B1. We have for $\theta < \pi/n$ and $n \geq 1$ an integer

$$\frac{1}{2}\left[\frac{1}{n}\mathrm{Cl}_2(2n\theta) - \mathrm{Cl}_2(2\theta)\right] = \sum_{j=1}^{\infty} \frac{\zeta(2j)}{j\pi^{2j}} \frac{(n^{2j}-1)}{(2j+1)}\theta^{2j+1} - \theta \ln n. \qquad (B.1)$$

This result gives several Corollaries, including

Corollary (i)

$$\frac{1}{2}\sum_{k=1}^{n-1}\mathrm{Cl}_2\left(2\theta + \frac{2\pi}{n}k\right) = \sum_{j=1}^{\infty} \frac{\zeta(2j)}{j\pi^{2j}} \frac{(n^{2j}-1)}{(2j+1)}\theta^{2j+1} - \theta \ln n, \qquad (B.2)$$

Corollary (ii)

$$\frac{1}{2}\left[\frac{1}{n}\mathrm{Cl}_2(2n\theta) - \mathrm{Cl}_2(2\theta)\right] = -\theta \ln n$$
$$+\frac{2\theta}{n}\int_0^{\infty} \frac{1}{x^2}[\sinh(nx) - n\sinh x]\frac{dx}{(e^{\pi x/\theta} - 1)}, \qquad (B.3)$$

Corollary (iii)

$$\frac{1}{2}\left[\frac{1}{n}\mathrm{Cl}_2(2n\theta) - \mathrm{Cl}_2(2\theta)\right] = -\theta \ln n$$
$$+\frac{\theta}{2\pi}\left[2n\theta\tanh^{-1}\left(\frac{n\theta}{\pi}\right) - 2\theta\tanh^{-1}\left(\frac{\theta}{\pi}\right) + \pi\ln\left(\frac{\pi^2 - n^2\theta^2}{\pi^2 - \theta^2}\right)\right]$$
$$+2\pi\int_1^{\infty}\left[\tanh^{-1}\left(\frac{\theta}{\pi x}\right) - \frac{1}{n}\tanh^{-1}\left(\frac{n\theta}{\pi x}\right)\right]P_1(x)dx. \qquad (B.4)$$

In the last equation, $P_1(x) = x - [x] - 1/2$ is the first periodized Bernoulli polynomial.

Proof. The Proposition is based upon the relation [1] (p. 75)

$$\ln\left(\frac{n\sin x}{\sin nx}\right) = \sum_{j=1}^{\infty} \frac{\zeta(2j)}{j\pi^{2j}}(n^{2j}-1)x^{2j}, \qquad |x| < \pi/n, \qquad (B.5)$$

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wherein we have used the relation between $\zeta(2j)$ and the Bernoulli numbers B_{2j}. (For more details, see the end of this Appendix.) With the series of (B.5) being boundedly convergent, we may integrate term-by-term over any finite interval avoiding the singularity at $x = \pi/n$. Doing so, integrating over $[0, \theta]$, and using the first integral representation for Cl_2 on the right side of (12) gives (B.1).

Corollary (i) follows from the multiplication formula for Cl_2 [14] (pp. 94, 253). Corollary (ii) uses a standard integral representation of the Riemann zeta function. With the interchange of summation and integration, with the integral being absolutely convergent, the Corollary follows.

Corollary (iii) uses the representation for Re $s > -1$,

$$\zeta(s) = \frac{1}{2} + \frac{1}{s - 1} - s \int_1^\infty \frac{P_1(x)}{x^{s+1}} dx. \qquad (B.6)$$

Again, the interchange of summation and integration is employed.

Remarks. In connection with (B.5) we may note the relation with the Chebyshev polynomials U_n of the second kind [12] (p. 1032),

$$U_{n-1}(\cos \phi) = \frac{\sin n\phi}{\sin \phi}. \qquad (B.7)$$

For $\theta = \pi/4$ and other values, Proposition B1 gives many relations involving the Catalan constant G. More generally, for θ a rational multiple of π, the results are expressible in terms of ψ', the trigamma function [11, 13]. If we let

$$r(\theta, n) \equiv \sum_{j=1}^\infty \frac{\zeta(2j)}{j\pi^{2j}} \frac{(n^{2j} - 1)}{(2j + 1)} \theta^{2j+1}, \qquad \theta \leq \pi/n, \qquad (B.8)$$

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we may write several simple examples:

$$r\left(\frac{\pi}{3}, 2\right) = \pi\left[\frac{1}{18}(\sqrt{3}\pi + 6\ln 2) - \frac{\psi'(1/3)}{4\sqrt{3}\pi}\right], \qquad (B.9a)$$

$$r\left(\frac{\pi}{3}, 3\right) = \pi\left[\frac{1}{27}(\sqrt{3}\pi + 9\ln 3) - \frac{\psi'(1/3)}{6\sqrt{3}\pi}\right], \qquad (B.9b)$$

$$r\left(\frac{\pi}{4}, 2\right) = -\frac{1}{2}G + \frac{\pi}{4}\ln 2, \qquad (B.10a)$$

$$r\left(\frac{\pi}{4}, 3\right) = -\frac{2}{3}G + \frac{\pi}{4}\ln 3, \qquad (B.10b)$$

$$r\left(\frac{\pi}{4}, 4\right) = -\frac{1}{2}G + \frac{\pi}{2}\ln 2, \qquad (B.10c)$$

and

$$r\left(\frac{\pi}{6}, 2\right) = \pi\left[\frac{1}{54}(2\sqrt{3}\pi + 9\ln 2) - \frac{\psi'(1/3)}{6\sqrt{3}\pi}\right], \qquad (B.11a)$$

$$r\left(\frac{\pi}{6}, 3\right) = \pi\left[\frac{1}{18}(\sqrt{3}\pi + 3\ln 3) - \frac{\psi'(1/3)}{4\sqrt{3}\pi}\right], \qquad (B.11b)$$

$$r\left(\frac{\pi}{6}, 4\right) = \pi\left[\frac{1}{108}(7\sqrt{3}\pi + 36\ln 2) - \frac{7\psi'(1/3)}{24\sqrt{3}\pi}\right], \qquad (B.11c)$$

$$r\left(\frac{\pi}{6}, 5\right) = \pi\left[\frac{1}{30}(2\sqrt{3}\pi + 5\ln 5) - \frac{\sqrt{3}\psi'(1/3)}{10\pi}\right], \qquad (B.11d)$$

$$r\left(\frac{\pi}{6}, 6\right) = \pi\left[\frac{1}{18}(\sqrt{3}\pi + 3\ln 2 + 3\ln 3) - \frac{\psi'(1/3)}{4\sqrt{3}\pi}\right]. \qquad (B.11e)$$

Finally, we supply a derivation of (B.5). We have

$$\frac{d}{dx}\ln\left(\frac{n\sin x}{\sin nx}\right) = \cot x - n\cot nx$$

$$= \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{(2k)!}(n^{2k} - 1)x^{2k-1}, \qquad \frac{n|x|}{\pi} < 1, \qquad (B.12)$$

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where we used a series representation for cot [12] (p. 35). Since

$$\frac{2^{2k}|B_{2k}|}{(2k)!} = \frac{2\zeta(2k)}{\pi^{2k}}, \qquad (B.13)$$

we have

$$\cot x - n \cot nx = 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} (n^{2k} - 1) x^{2k-1}. \qquad (B.14)$$

Integrating both sides of this relation gives (B.5).

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